Fixed Point Technique in Differential Equations

Md. Shariful Islam

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International Centre for Theoretical Physics
P.O. Box 586, I-34100 Trieste, Italy

Supervisor: Prof. Charles E. Chidume September 1997

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Contents

1 Introduction

Fixed point theory has proved to be a very powerful tool in the study of many problems arising in ordinary and partial differential equations; specially in problems related with the existence and uniqueness of solutions of differential equations.

Here we shall study some applications of *Contraction Mapping Principle* and *Schauder Fixed Point Theorem* to nonlinear ordinary differential equations. More precisely, we shall examine the roles of these theorems in the proofs of two of the generalized classical important theorems in ordinary differential equations; the *Picard-Lindelöf Theorem* and the *Peano Existence Theorem.* Indeed, we seek for the existence and uniqueness of solutions of the *Cauchy Problem*

$$x'(t) = f(t, x(t)), \quad x(t_0) = y_0, \;\; t, \; t_0 \in I \subset I\!R$$

where $x : I \mapsto Y$, Y is an arbitrary Banach space. Essentially we reduce the problems to the problems of the existence of fixed points for suitable integral operators defined on appropriate Banach spaces. The definitions of the integral operators will depend on the specific conditions of each problems.

Finally, we shall study the application of fixed point methods for studying the assymptotic equivalence between the nonlinear system

$$y'(x) = A(x)y + f(x, y)$$

and the linear system

$$z'(x) = A(x)z$$

where $A(x)$ is an $n \times n$ matrix of real-valued continuous functions for $x \geq a$, $a \in I\!R$, and $f : [a, \infty) \times I\!R^n$ is a continuous function. In fact, we shall prove that under certain assumptions the above two systems are asymptotically equivalent in the sense that the difference between two corresponding solutions under a homeomorphism tends to zero as x tends to infinity. For this, we shall once again use fixed point techniques to define the correspondence between the solutions of the two systems. We conclude our study with an

examination of how asymptotic equivalence can lead to stability of a given system of differential equations.

2 Generalized Picard-Lindelöf Theorem

Here we prove the classical theorem of Picard-Lindelöf in a generalized setting. Let Y be a real Banach space and let $x : [t_0 - c, t_0 + c] \to Y$ be a mapping into the Banach space Y. We seek for the existence and uniqueness of a solution of the initial value problem (in short, IVP)

$$x'(t) = f(t, x(t)), \quad x(t_0) = y_0 \tag{2.1}$$

where $y_0 \in Y$. Let the norm on Y be denoted by $\|.\|$, and let $X = C([t_0 - c, t_0 + c], Y)$, $0 < c < \infty$, be equipped with the norm $\|.\|_X$ defined by

$$\|x\|_X = \max_{t \in I_c} \|x(t)\|, \qquad I_c = [t_0 - c, t_0 + c].$$

Definition 2.1 *A function ϕ from a metric space (S, d) into itself is called a contraction mapping if there is a real number θ, $0 < \theta < 1$ such that*

$$d(\phi(s_1), \phi(s_2)) \leq \theta d(s_1, s_2)$$

for all s_1, $s_2 \in S$.

Theorem 2.1 *(Contraction Mapping Principle)*
If $\phi : (S, d) \to (S, d)$ is a contraction mapping and if (S, d) is a complete metric space, then ϕ has a unique fixed point; that is, there is a unique $s^* \in S$ such that $\phi(s^*) = s^*$.

Proof: The proof is quite standard and can be found in Chidume [3].

Lemma 2.1 *$(X, \|.\|_X)$ is a Banach Space.*

Proof: Let $< g_n >$ be a Cauchy sequence in $(X, \|.\|_X)$. This means

$$\max_{t \in I_c} \|g_k(t) - g_m(t)\| \to 0 \quad \text{as} \quad k, m \to \infty.$$

$$\Rightarrow \|g_k(t) - g_m(t)\| \to 0 \quad \text{as} \quad k, m \to \infty, \quad \text{for all} \quad t \in I_c.$$

For a fixed (but arbitrary) $t_1 \in I_c$ we get

$$\|g_k(t_1) - g_m(t_1)\| \to 0 \quad \text{as} \quad k, m \to \infty.$$

So, $< g_n(t_1) >$ is a Cauchy sequence in Y, which is a Banach space and hence complete. So $< g_n(t_1) >$ converges to some point, say $g(t_1) \in Y$. Hence for each $t \in I_c$, $< g_n(t) >$ converges to an element, say $g(t) \in Y$, i.e, $< g_n >$ converges point-wise to the function g. Now we show that this point-wise convergence is actually uniform on I_c.

Let $\epsilon > 0$ be given. Since $< g_n >$ is Cauchy, there exists $N \in I\!N$ such that $\|g_n - g_m\|_X < \epsilon/2$ for $n, m \geq N$. Let $t \in I_c$ and $n \geq N$. Then for m big enough we obtain

$$\begin{aligned} \|g_n(t) - g(t)\| &\leq \|g_n(t) - g_m(t)\| + \|g_m(t) - g(t)\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for all} \quad t \in I_c. \end{aligned}$$

$$\text{i.e,} \quad \|g_n(t) - g(t)\| < \epsilon \quad \text{for all} \quad t \in I_c \quad \textit{and} \quad n \geq N$$

$$\begin{aligned} &\Rightarrow \max_{t \in I_c} \|g_n(t) - g(t)\| \leq \epsilon \quad \text{for} \quad n \geq N \\ &\Rightarrow \|g_n - g\|_X \leq \epsilon \end{aligned}$$

Hence the convergence is uniform on I_c and consequently the limit function g is an element of X. Hence $(X, \|.\|_X)$ is a Banach space.

□

Now we state and prove the generalized Picard-Lindelöf theorem. The proof will be an application of the Contraction Mapping Principle.

Theorem 2.2 *Let Y be a real Banach space and let $t_0 \in I\!R$, $y_0 \in Y$, and*

$$Q_b := \{(t, y) \in I\!R \times Y : |t - t_0| \leq a, \|y - y_0\| \leq b\}$$

for fixed $a, b > 0$. Suppose $f : Q_b \to Y$ is continuous and

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad \textit{for all} \quad (t, x), (t, y) \in Q_b,$$

and

$$\|f(t,y)\| < K \quad \textit{for all} \quad (t,y) \in Q_b,$$

where $L \geq 0$ and $K > 0$ are fixed real numbers. Chose c such that $0 < c < a$ and $c < \min\{b/K, 1/L\}$. Then the

$$IVP: \quad x'(t) = f(t, x(t)), \quad x(t_0) = y_0$$

has exactly one continuously differentiable solution on the interval I_c.

Proof: Define the set $M := \{x \in X : \|x - y_0\|_X \leq b\}$ which is evidently a closed subset of X and hence M is complete as a metric space. Now we define an operator $T : M \to X$ as below:

$$(Tx)(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds \quad \text{for} \quad x \in M.$$

The reason of defining T on M is to make sure that the integral on the right side make sense because of the continuity of f on Q_b.
Now we check that the hypotheses of Contraction Mapping Principle are satisfied.

<u>(i) T maps M into M</u>

Let $x \in M$ be an arbitrary element. We show $Tx \in M$.

$$\begin{aligned} \|Tx - y_0\|_X &= \max_{t \in I_c} \| \int_{t_0}^{t} f(s, x(s))ds \| \\ &\leq K \max_{t \in I_c} |t - t_0| \\ &= Kc \leq b \end{aligned}$$

Hence $Tx \in M$.

<u>(ii) T is a contraction mapping</u>

Let $x, y \in M$. By using Lipschitz condition on f we get

$$\|Tx - Ty\|_X = \max_{t \in I_c} \| \int_{t_0}^{t} (f(s, x(s)) - f(s, y(s)))ds \|$$

$$
\begin{aligned}
&\leq L \max_{t \in I_c} \left| \int_{t_0}^{t} \|x(s)) - y(s)\| ds \right| \\
&\leq L \|x - y\|_X \max_{t \in I_c} |t - t_0| \\
&= Lc \|x - y\|_X
\end{aligned}
$$

Therefore, $\|Tx - Ty\|_X \leq Lc\|x - y\|_X$. According to our choice of c, $Lc < 1$, and consequently T is a contraction mapping.

Therefore, by Contraction Mapping Principle, T has a unique fixed point in M, i.e, $x(t) = (Tx)(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad t \in I_c$. Equivalently, the IVP: $x'(t) = f(t, x), \quad x(t_0) = y_0$ has a unique solution on I_c.

Remark: In general, T is not a contraction operator unless $Lc < 1$. Hence, if we apply the Contraction Mapping Principle to T, then we need to restrict the size of either the interval or L. An interesting remedy to this difficulty is outlined by Bielecki [2]. We describe his result below :

Define a new norm on X by

$$
\|x\|_1 := \max_{t \in I_c} \|x(t)\| e^{-K_0|t-t_0|},
$$

where K_0 is a fixed positive integer greater than L. We show that the two norms $\|.\|_X$ and $\|.\|_1$ are equivalent.

$$
\begin{aligned}
\|x\|_1 &= \max_{t \in I_c} \|x(t)\| e^{-K_0|t-t_0|} \\
&\leq \max_{t \in I_c} \|x(t)\| = \|x\|_X
\end{aligned}
$$

Again, since $t \in I_c$, $|t - t_0| \leq c$ which implies that $e^{-K_0|t-t_0|} \geq e^{-K_0 c}$. Consequently,

$$
\begin{aligned}
& \|x(t)\| e^{-K_0|t-t_0|} \geq \|x(t)\| e^{-K_0 c} \\
\Rightarrow \quad & \max_{t \in I_c} \|x(t)\| e^{-K_0|t-t_0|} \geq \max_{t \in I_c} \|x(t)\| e^{-K_0 c} \\
i.e, \quad & \|x\|_1 \geq e^{-K_0 c}.\|x\|_X
\end{aligned}
$$

Hence it follows that the two norms $\|.\|_X$ and $\|.\|_1$ are equivalent.

Since the two norms $\|.\|_X$ and $\|.\|_1$ are equivalent, it follows that $(X, \|.\|_1)$ is again a Banach space and M is a closed subset of X.
Now for $x, y \in M$, and $t \geq t_0$ we have

$$\begin{aligned}
\|(Tx)(t) - (Ty)(t)\| &\leq L \int_{t_0}^{t} \|x(s) - y(s)\| ds \\
&= L \int_{t_0}^{t} \|x(s) - y(s)\| e^{[K_0(s-t_0) - K_0(s-t_0)]} ds \\
&\leq L\|x - y\|_1 \int_{t_0}^{t} e^{K_0(s-t_0)} ds \\
&\leq L/K_0 \|x - y\|_1 [e^{K_0(t-t_0)} - 1] \\
&\leq L/K_0 \|x - y\|_1 e^{K_0(t-t_0)} \\
&\leq L/K_0 \|x - y\|_1 e^{K_0|t-t_0|}
\end{aligned}$$

For $t \leq t_0$ we have

$$\begin{aligned}
\|(Tx)(t) - (Ty)(t)\| &\leq -L \int_{t_0}^{t} \|x(s) - y(s)\| ds \\
&= -L \int_{t_0}^{t} \|x(s) - y(s)\| e^{[K_0(s-t_0) - K_0(s-t_0)]} ds \\
&= -L \int_{t_0}^{t} \|x(s) - y(s)\| e^{-K_0|s-t_0|} e^{-K_0(s-t_0)} ds \\
&= -L\|x - y\|_1 \int_{t_0}^{t} e^{-K_0(s-t_0)} ds \\
&= -L\|x - y\|_1 [\frac{e^{-K_0(s-t_0)}}{-K_0}]_{t_0}^{t} ds \\
&= L/K_0 \|x - y\|_1 [e^{-K_0(t-t_0)} - 1] \\
&= L/K_0 \|x - y\|_1 [e^{K_0|t-t_0|} - 1] \\
&\leq L/K_0 \|x - y\|_1 e^{K_0|t-t_0|}
\end{aligned}$$

Therefore,

$$\|(Tx)(t) - (Ty)(t)\| e^{-K_0|t-t_0|} \leq L/K_0 \|x - y\|_1$$

$$\Rightarrow \max_{t\in I_c} \|(Tx)(t) - (Ty)(t)\| e^{-K_0|t-t_0|} \leq L/K_0 \|x-y\|_1$$
$$\Rightarrow \|Tx - Ty\|_1 \leq L/K_0 \|x-y\|_1$$

Since $L/K_0 < 1$, T is indeed a contraction on M.

3 Generalized Peano Existence Theorem

By Schauder fixed point theorem, here we prove the generalized Peano Existence Theorem for the initial value problem:

$$x'(t) = f(t, x(t)), \quad x(t_0) = y_0 \tag{3.1}$$

for $x : [t_0 - c, t_0 + c] \to Y$, Y is a Banach space. In contrast to the generalized Picard-Lindelöf Theorem, we now require f to be compact, without assuming Lipschitz continuity.

At first we furnish necessary terminology and results that will be used in our proof.

Definition 3.1 *Let (X, ρ) be an arbitrary metric space and K be a subset of X. Then K is called relatively compact iff the closure of K is compact.*

Definition 3.2 *Let X and Y be Banach spaces, $T : D(T) \subset X \to Y$ an operator. T is called compact if (i) T is continuous and (ii) T maps bounded sets into relatively compact sets.*

Remark: For finite dimensional Banach spaces continuous and compact operators are the same whenever the domain $D(T)$ is closed. For if M is bounded, then $\bar{M}$, being closed and bounded subset of a finite dimensional space X, is compact. Consequently, $f(\bar{M})$, being the continuous image of $\bar{M}$, is compact. It remains to show that $f(\bar{M}) = \overline{f(M)}$. Since $M \subset \bar{M}$, obviously $f(M) \subset f(\bar{M}) \Rightarrow \overline{f(M)} \subset \overline{f(\bar{M})} = f(\bar{M})$. On the otherhand, because of continuity of f, we have $f(\bar{M}) \subset \overline{f(M)}$. Thus $f(\bar{M}) = \overline{f(M)}$.

Definition 3.3 *Let S be a nonempty subset (not neccessarily convex) of a normed linear space Y. The set*

$$coS = \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in S, \sum_{i=1}^{n} \lambda_i = 1, n = 0, 1, 2, ...\}$$

which consists of all convex combinations of elements of S is called the convex hull of S.
The closure of coS, denoted by $\overline{co}S$, is called the closed convex hull of S.

Lemma 3.1 *If $f : [a, b] \to Y$ is continuous, then*

$$(b-a)^{-1} \int_a^b f(t)dt \in \overline{co}\{f(t) : t \in [a, b]\}$$

Proof: We know, because of the continuity of f, that the integral $\int_a^b f(t)dt$ exists. By definition,

$$\int_a^b f(t)dt = \lim_{k\to\infty} S_{z_k},$$

where $S_{z_k} = \sum_{i=1}^{k} f(\bar{t}_i)(t_i - t_{i-1})$, $\bar{t}_i \in [t_{i-1}, t_i]$ is the partial sum corresponding to the partition $z_k : a = t_0 < t_1 < ... < t_k = b$ of $[a, b]$ and the sequence $< z_k >$ is such that $\Delta z_k := max_i(t_i - t_{i-1})$, the mesh of the partition, tends to zero as k tends to infinity. So, $(b-a)^{-1} \int_a^b f(t)dt$ exists, *i.e*, $(b-a)^{-1} \lim_{k\to\infty} S_{z_k}$ exists. *i.e*, $(b-a)^{-1} \lim_{k\to\infty}[\sum_{i=1}^{k} f(\bar{t}_i)(t_i - t_{i-1})]$ exists.

Since $f(\bar{t}_i) \in \{f(t) : t \in [a, b]\}$ and $\sum_{i=1}^{k} \frac{(t_i - t_{i-1})}{b-a} = 1$, it follows immediately, from the definition of convex hull of a set, that for each k

$$(b-a)^{-1} \sum_{i=1}^{k} f(\bar{t}_i)(t_i - t_{i-1}) \in co\{f(t) : t \in [a, b]\}$$

$$(b-a)^{-1} \lim_{k\to\infty}[\sum_{i=1}^{k} f(\bar{t}_i)(t_i - t_{i-1})] \in \overline{co}\{f(t) : t \in [a, b]\}$$

$$(b-a)^{-1}\int_a^b f(t)dt \in \overline{co}\{f(t) : t \in [a,b]\}$$

□

In proving the generalized Peano Existence Theorem we shall use the following theorems:

Theorem 3.1 *(Schauder fixed Point Theorem)*
Let M be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose $T : M \to M$ is a compact operator. Then T has a fixed point.

Theorem 3.2 *(Arzela-Ascoli Theorem)*
The set M in $C([a,b],Y)$ is relatively compact iff the set $\{f(x) : f \in M\}$ is relatively compact in Y for all $x \in [a,b]$; for every $x \in [a,b]$ and every $\epsilon > 0$ there is a $\delta(\epsilon, x) > 0$ which is independent of the function f, such that

$$\sup_{f \in M} \|f(x) - f(y)\|_Y < \epsilon$$

whenever $y \in [a,b]$ and $|x-y| < \delta(\epsilon, x)$.

Theorem 3.3 *(Mazur Theorem)*
If M is relatively compact in the Banach space Y, then the convex hull coM also is relatively compact and the closed convex hull $\overline{co}M$ is compact.

Theorem 3.4 *(Generalized Peano Existence Theorem)*
Let $t_0 \in R$, $y_0 \in Y$ and

$$Q_b = \{(t,y) \in R \times Y : |t - t_0| \leq a,\ \|y - y_0\| \leq b\},$$

for fixed numbers, $0 < a,\ b < \infty$. Suppose that $f : Q_b \to Y$ is compact and that $\|f(t,y)\| \leq K$ for all $(t,y) \in Q_b$ with fixed $K > 0$. We set $c = \min(a, b/K)$. Then the IVP (3.1) has a continuously differentiable solution on $I_c = [t_0 - c, t_0 + c]$.

Proof: Let us chose $X = C(I_c, Y)$, $0 < c < \infty$, with the norm

$$\|x\|_X = \max_{t \in I_c} \|x(t)\| \quad \text{and} \quad M = \{x \in X : \|x - y_0\|_X \leq b\}.$$

We have seen, in the proof of Picard-Lindelof theorem , that X, thus defined, is a Banach space and M is a nonempty (since $\tilde{x}(t) \equiv y_0$ is in M), closed, convex and bounded subset of X.

Now, after having a look into the Schauder Fixed Point theorem, the idea is to define an operator T on M into M, that will turn out to be a compact operator having a fixed point. That fixed point will be our desired solution.

Let's write the IVP (3.1) in its equivalent integral equation form:

$$x(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds, \forall t \in I_c.$$

In fact, the form of this integral equation helps us to define the right operator T on M. Let's define a map $T : M \to X$ by

$$(Tx)(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds, \ t \in I_c, \qquad (3.2).$$

First we show that T maps M into M indeed. Let $x \in M$ be arbitrary. It suffices to show $Tx \in M$. Now, $x \in M$ implies $\|x - y_0\|_X \leq b$, i.e, $\max_{t \in I_c} \|x(t) - y_0\| \leq b \Rightarrow \|x(t) - y_0\| \leq b$, for all $t \in I_c$. Let's set $(Tx)(t) = z(t)$. From (3.2) we get

$$\begin{aligned} \|z(t) - y_0\| &= \|\int_{t_0}^{t} f(s, x(s))ds\| \\ &\leq |\int_{t_0}^{t} \|f(s, x(s))\|ds| \\ &\leq K|t - t_0| \leq Kc \leq b \end{aligned}$$

Since $\|z(t) - y_0\| \leq b$, for all $t \in I_c$, we immediately obtain $\|z - y_0\|_X = \max_{t \in I_c} \|z(t) - y_0\| \leq b$. Hence T maps M into M. Now we show that T is

continuous and T maps bounded sets into relatively compact sets, i.e, T is a compact operator.

<u>T is continuous on M :</u>

To prove this, let $< x_n >$ be a sequence in M such that $x_n \to x$ as $n \to \infty$. We show $Tx_n \to Tx$. We remark that the convergence $x_n \to x$ with respect to the norm $\|.\|_X$ is uniform on I_c.

From (3.2), for $t \in I_c$, we get

$$\begin{aligned}
\|(Tx_n)(t) - (Tx)(t)\| &= \|\int_{t_0}^{t} [f(s, x_n(s)) - f(s, x(s))]ds\| \\
&\leq |\int_{t_0}^{t} \|f(s, x_n(s)) - f(s, x(s))\|ds| \\
&\leq |\int_{t_0}^{t} \sup_{s \in I_c} \|f(s, x_n(s)) - f(s, x(s))\|ds| \\
&\leq |t - t_0| \sup_{s \in I_c} \|f(s, x_n(s)) - f(s, x(s))\| \\
&\leq c \sup_{s \in I_c} \|f(s, x_n(s)) - f(s, x(s))\|
\end{aligned}$$

So, it follows that $\|Tx_n - Tx\|_X \leq c \ \sup_{s \in I_c} \|f(s, x_n(s)) - f(s, x(s))\|$. Now we claim : $\sup_{s \in I_c} \|f(s, x_n(s)) - f(s, x(s))\| \to 0$ as $n \to \infty$. Suppose $\sup_{s \in I_c} \|f(s, x_n(s)) - f(s, x(s))\| \to 0$ as $n \to \infty$, for a contradicton. Then, there would be an $\epsilon_0 > 0$ and a sequence, denoted by $< s_n >$ in $[t_0 - c, t_0 + c]$ for which

$$\|f(s_n, x_n(s_n)) - f(s_n, x(s_n))\| \geq \epsilon_0 \tag{3.3}$$

Since $[t_0 - c, t_0 + c]$ is compact and hence sequentially compact, so $< s_n >$ has a convergent subsequence again denoted by $< s_n >$ for brevity, such that $s_n \to s_0 \in I_c$ as $n \to \infty$, and

$$\begin{aligned}
\|x_n(s_n) - x(s_0)\| &\leq \|x_n(s_n) - x(s_n)\| + \|x(s_n) - x(s_0)\| \\
&\leq \|x_n - x\|_X + \|x(s_n) - x(s_0)\| \to 0 \ \ as \ \ n \to \infty.
\end{aligned}$$

The first term on the right tends to zero as $x_n \to x$, while the second one does so as $s_n \to s_0$ and x, being uniform limit of this sequence $< x_n >$

of continuous functions, is continuous. Hence, using the facts that $x_n(s_n) \to x(s_0)$, $x(s_n) \to x(s_0)$, $s_n \to s_0$ together with the continuity of f, we obtain that

$$f(s_n, x_n(s_n)) \to f(s_0, x(s_0))$$

and

$$f(s_n, x(s_n)) \to f(s_0, x(s_0)).$$

Consequently,

$$\|f(s_n, x_n(s_n)) - f(s_n, x(s_n))\| \to 0$$

which contradicts the inequality (3.3). Thus $\|Tx_n - Tx\|_X \to 0$ as desired.

Finally we show that $T : M \to M$ maps bounded sets into relatively compact sets. It suffices to prove that $T(M)$ is relatively compact. For if M' is any subset of M, then $\overline{T(M')} \subset \overline{T(M)}$ and $\overline{T(M')}$, being a closed set of a compact set, is compact. We'll use Arzela-Ascoli theorem to prove that $T(M)$ is relatively compact.

From equation (3.2) we get

$$\begin{aligned}
\|(Tx)(t_1) - (Tx)(t_2)\| &= \|z(t_1) - z(t_2)\| \\
&= \|\int_{t_0}^{t_1} f(s, x(s)) - \int_{t_0}^{t_2} f(s, x(s))ds\| \\
&= \|\int_{t_2}^{t_1} f(s, x(s))ds\| \\
&\leq K|t_1 - t_2|
\end{aligned}$$

for all $t_1,\ t_2 \in I_c$. Thus for every given $\epsilon > 0$, if we choose $\delta = \epsilon/K$, then

$$|t_1 - t_2| < \delta \Rightarrow \|z(t_1) - z(t_2)\| < \epsilon, \forall\, t_1,\ t_2 \in I_c \tag{3.4}$$

Since this inequality holds for all $z = Tx$ with arbitrary $x \in M$, we have

$$\sup_{z \in T(M)} \|z(t_1) - z(t_2)\| \leq \epsilon \ \text{ whenever } \ |t_1 - t_2| < \delta \tag{3.5}$$

Let's define a set $N(t)$ as below:

$$N(t) = y_0 + (t - t_0)\overline{co}\{f(s, x(s)) : s \in I_c,\ x \in M\}$$

Then it follows, from lemma 4.1, that

$$z(t) \in N(t), \quad \text{for all } z \in T(M), \quad \text{for all } t \in I_c \tag{3.6}$$

Since f is compact the set $\{f(s, x(s) : s \in I_c,\ x \in M\} \subset f(Q_b)$ is relatively compact and hence by Mazur Theorem $\overline{co}\{f(s, x(s)) : s \in I_c,\ x \in M\}$ is compact. Now, as the scalar multiplication and translation maps are continuous (indeed homeomorphisms), it follows that the set $N(t)$ is compact.

From (3.6) we have

$$\{z(t) : z \in T(M)\} \subset N(t)$$

$$\Rightarrow \overline{\{z(t) : z \in T(M)\}} \subset \overline{N(t)} = N(t).$$

Now $\overline{\{z(t) : z \in T(M)\}}$, being a closed subset of a compact set, is compact. So, from Arzela-Ascoli Theorem, it follows that $T(M)$ is relatively compact.

Now according to the Schauder fixed-point theorem T has a fixed point, say x, i.e,

$$x(t) = (Tx)(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds$$

which establishes the existence of a solution to our initial value problem.

□

4 Stability of Systems: Fixed Point Method

The stability of systems of nonlinear equations can be studied by different methods such as Liapunov's method, Olech's method, fixed point method, the method of logarithmic derivative and the method of invariant sets. We shall study, by using fixed point technique, the system

$$y' = A(x)y + f(x, y) \tag{4.1}$$

where $A(x)$ is an $n \times n$ matrix of real-valued continuous functions for $x \geq a$, $a \in R$, and $f : [a, \infty) \times R^n \to R^n$ is a continuous function. The system (4.1) is obtained by perturbing the system

$$z' = A(x)z \tag{4.2}$$

by a nonlinear term $f(.,.)$.

Definition 4.1 *Two systems of ordinary differential equation are asymptotically equivalent if there is a correspondence between the solutions of the two systems such that the difference between two corresponding solutions tends to zero as x tends to infinity.*

In this section we shall study asymptotic equivalence between the two systems (4.1) and (4.2). We shall also see how asymptotic equivalence can lead to stability.

Theorem 4.1 *(**On Asymptotic Equivalence for Bounded Solution**) Let $A(x)$ be an $n \times n$ matrix continuous for $x \geq a$, a is a constant, $Z(x)$ a fundamental matrix of the system (4.2) and $f : [a, \infty) \times R^n \to R^n$ a continuous function such that*

$$\int_a^\infty \|f(t, 0)\| dt < \infty \quad \textit{and} \quad \|f(x, y) - f(x, \bar{y})\| \leq h(x)\|y - \bar{y}\|$$

with $h(t)$ bounded on $[a,\infty)$ and $\int_a^\infty h(t)dt < \infty$. If there are two supplementary projections P_1 and P_2 of R^n and a constant $K > 0$ such that

$$\|Z(x)P_1Z^{-1}(t)\| \leq K \quad for \quad a \leq t \leq x$$

$$\|Z(x)P_2Z^{-1}(t)\| \leq K \quad for \quad a \leq x \leq t$$

$$\lim_{x\to+\infty} Z(x)P_1 = 0,$$

then there is a one-one correspondence S between the set of bounded solutions of (4.2) and the set of bounded solutions of (4.1); the correspondence is a homeomorphism in the topology of uniform convergence on $[a,\infty)$ and satisfies the relations

$$\lim_{x\to\infty} \|z(x) - Sz(x)\| = 0,$$

$$z(c) - Sz(c) = \int_c^\infty Z(c)P_2Z^{-1}(t)f(t,Sz(t))dt$$

for every bounded solution z of (4.2) with $c \geq a$ sufficiently large.

Proof: Since $\int_a^\infty h(t)dt < \infty$, we have

$$\lim_{c\to\infty} 2K \int_c^\infty h(t)dt = 0.$$

So, we can choose $c \geq a$ such that

$$2K \int_c^\infty h(t)dt < 1 \tag{4.3}$$

By virtue of the uniqueness and continuous dependence of solutions on compact intervals in the Lipschitz case, it is enough to prove the theorem for solutions on the interval $[c,\infty)$. More explicitly, we need to show that the correspondence S between the set of bounded solutions of (4.2), say $X_1([a,\infty))$, and the set of bounded solutions of (4.1), say $X_2([a,\infty))$, is a homeomorphism. By using the uniqueness and continuous dependence of solutions on compact intervals, we'll later show that $X_1([a,\infty))$ and $X_1([c,\infty))$ and

$X_2([a,\infty))$ and $X_2([c,\infty))$ are homeomorphic spaces. Hence it suffices to prove the existence of a map which is a homeomorphism between $X_1([c,\infty))$ and $X_2([c,\infty))$.

Let's consider the Banach space X of bounded and continuous functions from $[c,\infty)$ into R^n with the sup norm $\|.\|_\infty$. Let z be a bounded solution of (4.2) and hence $z \in X$. Corresponding to z we define a transformation T_z on X as below:

$$\begin{aligned}(T_z y)(x) = z(x) \quad &+ \quad \int_c^x Z(x)P_1 Z^{-1}(t)f(t,y(t))dt \\ &- \quad \int_x^\infty Z(x)P_2 Z^{-1}(t)f(t,y(t))dt \ldots\ldots\ldots\ldots (4.4)\end{aligned}$$

Now we verify that T_z satisfies the hypothesis on contraction mapping theorem.

<u>(i) T maps X into X</u>

Clearly, the first term on the right of equation (4.4) is bounded. On the other hand, by using the hypothesis

$$\|f(t,y) - f(t,\bar{y})\| \leq h(t)\|y - \bar{y}\|$$

we have

$$\begin{aligned}\|\int_c^x Z(x)P_1 Z^{-1}(t)f(t,y(t))dt\| &\leq \int_c^x \|Z(x)P_1 Z^{-1}(t)\|\|f(t,y(t))\|dt \\ &\leq K\int_c^x \|f(t,y(t))\|dt \\ &\leq K\left[\int_c^x (\|f(t,0)\| + h(t)\|y\|_\infty)dt\right] \\ &\leq K\left[\int_c^\infty (\|f(t,0)\| + h(t)\|y\|_\infty)dt\right] \\ &\leq K\int_c^\infty \|f(t,0)\|dt + K\|y\|_\infty)\int_c^\infty h(t)dt\end{aligned}$$

and analogously

$$\|\int_x^\infty Z(x)P_2Z^{-1}(t)f(t,y(t))dt\| \leq K\int_c^\infty \|f(t,0)\|dt + K\|y\|_\infty)\int_c^\infty h(t)dt$$

So, by using the fact that $y \in X$, i.e, y is bounded, and the hypotheses: $\int_a^\infty \|f(t,0)\|dt < \infty$ and $\int_a^\infty h(t)dt < \infty$, we conclude that the last two terms in (4.4) are bounded on $[c,\infty)$. Hence T maps X into itself.

<u>(ii) T is a contraction</u>

Let y_1 and y_2 be in X. We compute:

$$\begin{aligned}
\|(T_zy_1)(x) - (T_zy_2)(x)\| &\leq \|\int_c^x Z(x)P_1Z^{-1}(t)[f(t,y_1(t)) - f(t,y_2(t))]\|dt \\
&+ \|\int_x^\infty Z(x)P_2Z^{-1}(t)[f(t,y_1(t)) - f(t,y_2(t))]\|dt \\
&\leq \int_c^x \|Z(x)P_1Z^{-1}(t)\|\|f(t,y_1(t)) - f(t,y_2(t))\|dt \\
&+ \int_x^\infty \|Z(x)P_2Z^{-1}(t)\|\|f(t,y_1(t)) - f(t,y_2(t)\|dt \\
&\leq \int_c^x Kh(t)\|y_1 - y_2\|_\infty dt + \int_x^\infty Kh(t)\|y_1 - y_2\|_\infty dt \\
&\leq 2K\|y_1 - y_2\|_\infty \int_c^\infty h(t)dt
\end{aligned}$$

Since, from (4.3), $2K\int_c^\infty h(t)dt < 1$, after taking supremum as x runs through $[c,\infty)$, it follows that T_z is a contraction mapping on X. Consequently, by contraction mapping principle, T_z has a unique fixed point y_z. We now define S by setting $Sz = y_z$. Because of uniqueness of y_z, the map S is well-defined. We now show that this y_z is a solution of (4.1). We have

$$\begin{aligned}
y_z(x) = (T_zy_z)(x) &= z(x) + \int_c^x Z(x)P_1Z^{-1}(t)f(t,y_z(t))dt \\
&- \int_x^\infty Z(x)P_2Z^{-1}(t)f(t,y_z(t))dt
\end{aligned}$$

Differentiating with respect to x we get

$$\begin{aligned}
y_z'(x) &= z'(x) + Z(x)P_1Z^{-1}(x)f(x, y_z(x)) + Z'(x)\int_c^x P_1Z^{-1}(t)f(t, y_z(t))dt \\
&- Z'(x)\int_x^\infty P_2Z^{-1}(t)f(t, y_z(t))dt \\
&- Z(x)[\frac{d}{dx}\int_x^\infty P_2Z^{-1}(t)f(t, y_z(t))dt] \\
&= A(x)z(x) + Z(x)P_1Z^{-1}(x)f(x, y_z(x)) + Z'(x)\int_c^x P_1Z^{-1}(t)f(t, y_z(t))dt \\
&- Z'(x)\int_x^\infty P_2Z^{-1}(t)f(t, y_z(t))dt \\
&- Z(x)[\frac{d}{dx}\{\int_c^\infty P_2Z^{-1}(t)f(t, y_z(t))dt - \int_c^x P_2Z^{-1}(t)f(t, y_z(t))dt \\
&= A(x)z(x) + Z(x)[P_1 + P_2]Z^{-1}(x)f(x, y_z(x)) \\
&+ A(x)Z(x)\int_c^x P_1Z^{-1}(t)f(t, y_z(t))dt \\
&- A(x)Z(x)\int_x^\infty P_2Z^{-1}(t)f(t, y_z(t))dt \\
&= A(x)[z(x) + \int_c^x Z(x)P_1Z^{-1}(t)f(t, y_z(t))dt \\
&- \int_x^\infty Z(x)P_2Z^{-1}(t)f(t, y_z(t))dt] + f(x, y_z(x)) \\
&= A(x)y_z(x) + f(x, y_z(x)).
\end{aligned}$$

Hence y_z is a solution of (4.1). So, S maps bounded solution of (4.2) into bounded solution of (4.1). Now we prove S is one-one, onto and bicontinuous; that is a homeomorphism between $X_1([c, \infty))$ and $X_2([c, \infty))$.

<u>S is onto</u>

Let y be a bounded solution of (4.1). Let's define z as below:

$$\begin{aligned}
z(x) = y(x) &- \int_c^x Z(x)P_1Z^{-1}(t)f(t, y(t))dt \\
&+ \int_x^\infty Z(x)P_2Z^{-1}(t)f(t, y(t))dt \text{.....}(4.5)
\end{aligned}$$

We show indeed $z \in X_1([c,\infty))$.

<u>z satisfies (4.2)</u>

Differentiating (4.5) with respect to x we get

$$\begin{aligned}
z'(x) = y'(x) \; &- \; Z'(x)\int_c^x P_1 Z^{-1}(t) f(t,y(t))dt \\
&- \; Z(x)P_1Z^{-1}(x)f(x,y(x)) + Z'(x)\int_x^\infty P_2Z^{-1}(t)f(t,y(t))dt \\
&+ \; Z(x)[\frac{d}{dx}\{\int_c^\infty P_2Z^{-1}(t)f(t,y(t))dt - \int_c^x P_2Z^{-1}(t)f(t,y(t))dt\}] \\
&= \; A(x)y(x) + f(x,y) - A(x)Z(x)\int_c^x P_1Z^{-1}(t)f(t,y(t))dt \\
&- \; Z(x)P_1Z^{-1}(x)f(x,y(x)) + A(x)Z(x)\int_x^\infty P_2Z^{-1}(t)f(t,y(t))dt \\
&- \; Z(x)P_2Z^{-1}(x)f(x,y(x)) \\
&= \; A(x)[y(x) - \int_c^x Z(x)P_1Z^{-1}(t)f(t,y(t))dt \\
&+ \; \int_x^\infty Z(x)P_2Z^{-1}(t)f(t,y(t))dt] \\
&= \; A(x)z(x)
\end{aligned}$$

<u>z is bounded on $[c,\infty)$</u>

From (4.5) we get

$$\begin{aligned}
\|z(x)\| \le \|y(x)\| \; &+ \; \int_c^x \|Z(x)P_1Z^{-1}(t)\|\|f(t,y(t))\|dt \\
&+ \; \int_x^\infty \|Z(x)P_2Z^{-1}(t)\|\|f(t,y(t))\|dt \\
&\le \; \|y(x)\| + \int_c^x K[\|f(t,y(t)) - f(t,0)\| + \|f(t,0)\|]dt \\
&+ \; \int_x^\infty K[\|f(t,y(t)) - f(t,0)\| + \|f(t,0)\|]dt \\
&\le \; \|y(x)\| + \int_c^x Kh(t)\|y(t)\|dt + \int_c^x K\|f(t,0)\|dt \\
&+ \; \int_x^\infty Kh(t)\|y(t)\|dt + \int_x^\infty K\|f(t,0)\|dt
\end{aligned}$$

Taking supremum as $x \in [c, \infty)$ and using the fact that y is bounded solution of (4.1) and the hypotheses: $\int_a^\infty \|f(t,0)\|dt < \infty$ and $\int_a^\infty h(t)dt < \infty$, it follows that $\|z\|_\infty < \infty$.

Thus indeed $z \in X_1([c, \infty))$. Hence, from (4.5) and the uniqueness of the fixed point of T_z, it follows that $Sz = y$. Thus S is onto.

<u>S is one-one</u>

Let $Sz_1 = y = T_{z_1}(y)$ and $Sz_2 = y = T_{z_2}(y)$. Now from (4.4) we obtain

$$\|z_1 - z_2\| = \|T_{z_1}y - T_{z_2}y\|.$$

As a result, $Sz_1 = Sz_2$ immediately implies $z_1 = z_2$.

<u>S is continuous</u>

We have $Sz_1 = y_1 = T_{z_1}Sz_1$ and $Sz_2 = y_2 = T_{z_2}Sz_2$. Now

$$\begin{aligned}
\|Sz_1 - Sz_2\|_\infty &= \|T_{z_1}Sz_1 - T_{z_2}Sz_2\|_\infty \\
&= \|T_{z_1}y_1 - T_{z_2}y_2\|_\infty \\
&= \|T_{z_1}y_1 - T_{z_1}y_2 + T_{z_1}y_2 - T_{z_2}y_2\|_\infty \\
&\leq \|T_{z_1}y_1 - T_{z_1}y_2\|_\infty + \|T_{z_1}y_2 - T_{z_2}y_2\|_\infty \\
&\leq \alpha\|y_1 - y_2\|_\infty + \|z_1 - z_2\|_\infty
\end{aligned}$$

$$\begin{aligned}
i.e, \quad & \|y_1 - y_2\|_\infty \leq \alpha\|y_1 - y_2\|_\infty + \|z_1 - z_2\|_\infty \\
i.e, \quad & \|y_1 - y_2\|_\infty \leq \frac{1}{1-\alpha}\|z_1 - z_2\|_\infty \\
i.e, \quad & \|Sz_1 - Sz_2\|_\infty \leq \frac{1}{1-\alpha}\|z_1 - z_2\|_\infty,
\end{aligned}$$

where $\alpha = 2K\int_c^\infty h(t)dt < 1$. Thus, for given $\epsilon > 0$, we can choose $\delta \in (0, \epsilon(1-\alpha))$ such that

$$\|z_1 - z_2\|_\infty < \delta \Rightarrow \|Sz_1 - Sz_2\|_\infty < \epsilon$$

Hence S is continuous.

<u>S^{-1} is continuous</u>

Let $S^{-1}(y_1) = z_1$, $S^{-1}(y_2) = z_2$. From (4.5) we get

$$\begin{aligned}
\|z_1(x) - z_2(x)\| &\leq \|y_1(x) - y_2(x)\| + \int_c^x K\|f(t, y_1(t)) - f(t, y_2(t))\|dt \\
&+ \int_x^\infty K\|f(t, y_1(t)) - f(t, y_2(t))\|dt \\
&\leq \|y_1(x) - y_2(x)\| + \int_c^x Kh(t)\|y_1(t) - y_2(t)\|dt \\
&+ \int_x^\infty Kh(t)\|y_1(t) - y_2(t)\|dt \\
&\leq \|y_1(x) - y_2(x)\| + \int_c^x Kh(t) \sup_{t\in[c,\infty)} \|y_1(t) - y_2(t)\|dt \\
&+ \int_x^\infty Kh(t) \sup_{t\in[c,\infty)} \|y_1(t) - y_2(t)\|dt \\
&\leq \|y_1(x) - y_2(x)\| + K\|y_1 - y_2\|_\infty \int_c^x h(t)dt \\
&+ K\|y_1 - y_2\|_\infty \int_x^\infty h(t)dt \\
&\leq \|y_1 - y_2\|_\infty (1 + K\int_c^\infty h(t)dt)
\end{aligned}$$

Taking supremum as $x \in [c, \infty)$ we get

$$\|z_1 - z_2\|_\infty \leq \|y_1 - y_2\|_\infty (1 + 2K\int_c^\infty h(t)dt).$$

$$\Rightarrow \quad \|S^{-1}y_1 - S^{-1}y_2\|_\infty \leq \|y_1 - y_2\|_\infty (1 + 2K\int_c^\infty h(t)dt).$$

Hence S^{-1} is continuous.

From (4.5) we obtain

$$z(c) = y(c) + \int_c^\infty Z(c)P_2Z^{-1}(t)f(t, y(t))dt$$

By putting $Sz = y$ we get the relation

$$z(c) - Sz(c) = \int_c^\infty Z(c)P_2Z^{-1}(t)f(t, Sz(t))dt \tag{4.6}$$

It remains to show that

$$\lim_{x\to\infty} \|z(x) - Sz(x)\| = 0$$

Again from (4.5), for $x \geq x_1 \geq c$, we have

$$\begin{aligned}
\|z(x) - Sz(x)\| &\leq \int_c^x \|Z(x)P_1Z^{-1}(t)f(t, Sz(t))\|dt \\
&+ \int_x^\infty \|Z(x)P_2Z^{-1}(t)f(t, Sz(t))\|dt \\
&= \int_c^{x_1} \|Z(x)P_1Z^{-1}(t)f(t, Sz(t))\|dt \\
&+ \int_{x_1}^x \|Z(x)P_1Z^{-1}(t)f(t, Sz(t))\|dt \\
&+ \int_x^\infty \|Z(x)P_2Z^{-1}(t)f(t, Sz(t))\|dt \\
&\leq \|Z(x)P_1\| \int_c^{x_1} \|Z^{-1}(t)f(t, Sz(t))\|dt \\
&+ K\int_{x_1}^\infty \|f(t, Sz(t))\|dt + K\int_{x_1}^\infty \|f(t, Sz(t))\|dt \\
&= \|Z(x)P_1\| \int_c^{x_1} \|Z^{-1}(t)f(t, Sz(t))\|dt \\
&+ 2K\int_{x_1}^\infty \|f(t, Sz(t))\|dt \ldots\ldots(4.7)
\end{aligned}$$

Now, from the inequality

$$\|f(x, Sz)\| \leq h(x)\|y\| + \|f(x, 0)\|,$$

after using the hypotheses $\int_a^\infty \|f(t,0)\|dt < \infty$ and $\int_a^\infty h(t)dt < \infty$, we find that $2K\int_a^\infty \|f(t, Sz(t))\|dt < \infty$. Consequently, since

$$\lim_{x_1\to\infty} 2K\int_{x_1}^\infty \|f(t, Sz(t))\| = 0,$$

for given $\epsilon > 0$ we can determine $x_1 \geq c$ such that the second integral in (4.7) is less than $\epsilon/4$. With x_1 thus fixed, the first term on the right hand

side of (4.7) tends to zero as x tends to infinity by virtue of the hypothesis

$$\lim_{x\to\infty} Z(x)P_1 = 0.$$

Thus we conclude that

$$\lim_{x\to\infty} \|z(x) - Sz(x)\| = 0$$

and the theorem is completely proved.

□

Remark: Here, we show that $X_1([a,\infty))$ and $X_1([c,\infty))$ are homeomorphic spaces. The proof for $X_2([a,\infty))$ and $X_2([c,\infty))$ are alomost similar. We need the following theorem in our proof.

Theorem 4.2 *(On Continuous Dependence of Solutions)*
Let the function $F(t,x)$ be continuous in the set

$$B = \{(t,x) : t_0 \le t \le t_0 + a,\ \|x - x_0\| \le b\}$$

and satisfy the Lipschitz condition

$$\|F(t,x_1) - F(t,x_2)\| \le K\|x_1 - x_2\| \quad for \quad (t,x_1),\ (t,x_2) \in B.$$

Then, $x_n \to x_0$ implies $x(t,t_0,x_0) \to x(t,t_0,x_0)$ uniformly for $t \in [t_0, t_0 + a]$, where $x(t) = x(t,t_0,x_0)$ denotes a solution of $x' = F(t,x)$ through the initial point (t_0,x_0).

Proof: Let $x(t,t_0,x_n)$ and $x(t,t_0,x_n)$ be two solutions of $x' = F(t,x)$ through the initial points (t_0,x_n) and (t_0,x_0) respectively. Then, we have

$$x(t,t_0,x_n) = x_n + \int_{t_0}^{t} F(s,x(s,t_0,x_n))ds,$$

and

$$x(t,t_0,x_0) = x_0 + \int_{t_0}^{t} F(s,x(s,t_0,x_0))ds.$$

Using the Lipschitz condition, we obtain, for $t \geq t_0$,

$$\|x(t,t_0,x_n) - x(t,t_0,x_0)\| \leq \|x_n - x_0\| + \int_{t_0}^{t} K\|x(s,t_0,x_n) - x(s,t_0,x_0)\|ds.$$

The application of Gronwall inequality yields

$$\|x(t,t_0,x_n) - x(t,t_0,x_0)\| \leq \|x_n - x_0\|e^{Ka}$$

and this, in turn, implies the result.

□

Now we prove that the two spaces $X_1([a,\infty))$ and $X_1([c,\infty))$ are homeomorphic.

Define a map

$$T : X_1([a,\infty)) \mapsto X_1([c,\infty))$$

by $T(f) := f|_{[c,\infty)}$ for $f \in X_1([a,\infty))$. We show that the map thus defined is a homeomorphism between the two spaces $X_1([a,\infty))$ and $X_1([c,\infty))$.

<u>T is one-one</u>

Let $f,\ g \in X_1([a,\infty))$. Now $T(f) = T(g) \Leftrightarrow f|_{[c,\infty)} = g|_{[c,\infty)}$. In particular $f(c) = g(c) = z_0$, say. Since both f and g are solutions of the Cauchy problem

$$z' = A(x)z, \quad z(c) = z_0$$

on $[a,\infty)$, by uniqueness theorem we get $f \equiv g$ on $[a,\infty)$.

<u>T is onto</u>

Let $\phi \in X_1([c,\infty))$. Since $A(x)$ is continuous on $[a,\infty)$, the solution of the Cauchy problem

$$z' = A(x)z, \quad z(c) = \phi(c),$$

call it ψ, is a global solution; that is ψ is defined on $[a,\infty)$. Now its restriction on $[c,\infty)$ is also a solution of the same Cauchy problem. Since $\psi(c) = \phi(c)$, by uniqueness theorem it follows that $T(\psi) = \psi|_{[c,\infty)} = \phi$. Hence T is onto.

T is continuous

Let $< f_n >$ be a sequence in $X_1([a,\infty))$ converging to some element $f \in X_1([a,\infty))$. That is, $f_n \to f \Rightarrow \|f_n - f\|_\infty < \epsilon$. Now

$$\|Tf_n - Tf\|_\infty = \|f_n|_{[c,\infty)} - f|_{[c,\infty)}\|_\infty < \epsilon$$

since we're taking supremum of $f_n - f$ on a smaller set $[c,\infty)$. Hence T is continuous.

T^{-1} is continuous

Let g_n be a sequence in $X_1([c,\infty))$ converging to some element $g \in X_1([a,\infty))$. Then, in particular, $g_n(c) \to g(c)$. We need to show that $T^{-1}g_n \to T^{-1}g$. Let $\tilde{g_n}$ be the sequence of solutions of the Cauchy problem

$$z' = A(x)z, \quad z(c) = g_n(c)$$

defined on $[a,\infty)$. Then $T\tilde{g_n} := \tilde{g_n}|_{[c,\infty)}$ is a solution of

$$z' = A(x)z, \quad z(c) = g_n(c)$$

defined on $[c,\infty)$. So, by uniqueness of solutions,

$$T\tilde{g_n} = g_n \quad \text{on} \quad [c,\infty).$$

In the same way if $\tilde{g}$ is the solution of the Cauchy problem

$$z' = A(x)z, \quad z(c) = g(c)$$

defined on $[a,\infty)$, then $T\tilde{g} = g$. Thus we have $T^{-1}g_n = \tilde{g_n}$ and $T^{-1}g = \tilde{g}$. Now, let $x \in [a, c+\delta]$. Since $\tilde{g_n}(c) = g_n(c) \to g(c) = \tilde{g}(c)$, by the continuous dependence of solutions on the initial data, we have that $\tilde{g_n}$ approach $\tilde{g}$ uniformly on $[a, c+\delta]$ as $\tilde{g_n}(c)$ get close $\tilde{g}(c)$. That is, for give $\epsilon > 0$, we can find $N \in I\!N$ such that for all $x \in [a, c+\delta]$ and for all $n \geq N$

$$\|\tilde{g_n}(x) - \tilde{g}(x)\| < \epsilon$$

On the other hand, for $x \in (c+\delta, \infty)$ we have

$$\begin{aligned} \|\tilde{g_n}(x) - \tilde{g}(x)\| &= \|\tilde{g_n}|_{[c,\infty)}(x) - \tilde{g}|_{[c,\infty)}(x)\| \\ &= \|(T\tilde{g_n})(x) - (\tilde{g})(x)\| \\ &= \|g_n(x) - g(x)\| \end{aligned}$$

Since $g_n \to g$ in $[c, \infty)$ uniformly, given $\epsilon > 0$ there exists $\tilde{N} \in I\!N$ such that for $n \geq \tilde{N}$, and for all $x \in [c, \infty)$

$$\|g_n(x) - g(x)\| < \epsilon$$

Now by taking $N' = \max\{N, \tilde{N}\}$, for all $n \geq N'$ and every $x \in [a, \infty)$ we get

$$\|\tilde{g_n}(x) - \tilde{g}(x)\| < \epsilon$$

That is, $\tilde{g_n} \to \tilde{g}$ uniformly on $x \in [a, \infty)$. Cosequently, $T^{-1}g_n \to T^{-1}g$ uniformly on $x \in [a, \infty)$ as required.

Before proving the first corollary of the above theorem we need few results. The first one is on the Jordan canonical form of an arbitrary $n \times n$ constant matrix A.

Theorem 4.3 *(Bellman [1]) Let $\lambda_1, \lambda_2, ...\lambda_r$ be the characteristic roots of A with multiplicities $k_1, k_2, ..., k_r$ respectively, such that $k_1 + k_2 + ... + k_r = n$. Then, there exists an invertible matrix T such that*

$$T^{-1}AT = \begin{pmatrix} L_{k_1}(\lambda_1) & 0 & ... & 0 \\ 0 & L_{k_2}(\lambda_2) & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ 0 & 0 & ... & L_{k_r}(\lambda_r) \end{pmatrix},$$

where $L_{k_j}(\lambda_j)$ is a $k_j \times k_j$ matrix given by

$$L_{k_j}(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda_j & 1 & 0 & 0 & 0 \\ . & . & ... & . & . & . \\ . & . & ... & . & . & . \\ . & . & . & ... & \lambda_j & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda_j \end{pmatrix},$$

and $L_1(\lambda_j) = \lambda_j$. In particular, if the characteristic roots of A are distinct, then there exists an invertible matrix T such that

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ . & . & \dots & . & . & . \\ . & . & \dots & . & . & . \\ . & . & \ddots & \dots & \lambda_j & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix},$$

The second result that will be used in proving the corollary of our theorem is as below.

Lemma 4.1 *(i) Let $f(t) = t^a e^{-bt}$ for some positive constants a and b, let c be a positive number less than b. Then, there exists a positive constant K such that $|f(t)| \leq Ke^{-ct}, \quad 0 \leq t \leq \infty$.*

(ii) If all the eigenvalues of A have negative real part, then there exists positive constants K and α such that $|(e^{At})_{ij}| \leq Ke^{-\alpha t}$ for $1 \leq i,\ j \leq n$.

Proof: (i) Since

$$\lim_{t\to\infty} f(t)/e^{-ct} = \lim_{t\to\infty} t^a/e^{(b-c)t} = 0,$$

for any $\epsilon > 0$ there exists $t_0 > 0$ such that if $t > t_0$ then $|f(t)/e^{-ct}| < \epsilon$, i.e, $|f(t)| < \epsilon e^{-ct}$ for all $t > t_0$.
Now, let's consider $g(t) = f(t)/e^{-ct}$. g, being composition of two continuous functions with $e^{-ct} = 0$ for all $t \geq 0$, is continuous on $[0, \infty)$ and hence it is continuous on $[0, t_0]$. Since $[0, t_0]$ is compact, g is bounded there. Let

$$|g(t)| \leq M_0 \quad \text{for all} \quad t \in [0, t_0]$$

Taking $K = \max\{\epsilon, M_0\}$ we get the desired result.

(ii) By using (i), the result immediately follows from the fact that each component of e^{At} is a finite linear combination of functions of the form $q(t)e^{\lambda t}$, where $q(t)$ is a polynomial.

□

Lemma 4.2 *Let A be a constant $n \times n$ matrix whose eigenvalues have non-positive real part; suppose that the eigenvalues with real part zero are simple. Let $f : [a, \infty) \times R^n \to R^n$ be continuous and such that*

$$\int_a^\infty \|f(t,0)\|dt < \infty \quad \textit{and} \quad \|f(x,y) - f(x,\bar{y})\| \leq h(x)\|y - \bar{y}\|$$

with $h(t)$ bounded on $[a, \infty)$ and $\int_a^\infty h(t)dt < \infty$. Then all the solutions of the two systems

$$y' = Ay + f(x,y) \tag{4.1c}$$

$$z' = Az \tag{4.2c}$$

are bounded.

Proof: Let's prove the result for (4.2c) first. Let $\lambda_j = ib_j$, $j = 1, 2, ..., r$, be the simple eigenvalues of A with zero real parts. Then, a solution of (4.2c) corresponding to $\lambda_j = ib_j$ is of the following form.

$$z_j(t) = (z_{1j}(t), z_{2j}(t), ..., z_{nj}(t)) = e^{ib_j t}v,$$

where v is the corresponding constant eigenvector. Clearly $z_j(t)$ is bounded on $[a, \infty)$; let M_0 be the bound. Therefore,

$$|z_{ij}(t)| \leq M_0, \quad j = 1, 2, ..., r, \quad i = 1, 2, ..., n \tag{4.8}$$

Now, let $\lambda_j = \alpha_j + i\beta_j$, $j = r+1, ..., n$ be the eigenvalues with $\alpha_j < 0$. Then, by the above lemma , we have positive constants K and α such that

$$|z_{ij}(t)| \leq Ke^{-\alpha t}, \quad \text{for} \quad j = r+1, ..., n, \quad i = 1, 2, ..., n \tag{4.9}$$

Now, every solution $z = \psi(t)$ of $z' = Az$ is of the form $\psi(t) = e^{At}\psi(0)$. Let $z_{ij}(t)$ be the $ij - th$ element of the matrix e^{At}, and let $\psi_1^0, \psi_2^0, ..., \psi_n^0$ be the components of $\psi(0)$. Then, the $i - th$ component of $\psi(t)$ is:

$$\psi_i(t) = z_{i1}(t)\psi_1^0 + ... + z_{in}(t)\psi_n^0 = \sum_{j=1}^{n} z_{ij}(t)\psi_j^0.$$

Therefore,

$$\begin{aligned} |\psi_i(t)| &= |\sum_{j=1}^{r} z_{ij}(t)\psi_j^0 + \sum_{j=r+1}^{n} z_{ij}(t)\psi_j^0| \\ &\leq M_0 \sum_{j=1}^{r} |\psi_j^0| + Ke^{-\alpha t} \sum_{j=r+1}^{n} |\psi_j^0| \end{aligned}$$

Since

$$\lim_{t\to\infty} e^{-\alpha t} = 0,$$

$e^{-\alpha t}$ is bounded function on any interval $[a, \infty)$. Hence it follows that all the solutions of (4.2c) are bounded.

Now we prove that all the solutions of (4.1c) are bounded. Treating $f(x, y)$ as an inhomogeneous term and applying the variation of constants formula we get that every solution $y(x)$ of (4.1c) is expressed by the following integral equation:

$$y(x) = z(x) + \int_a^x \Phi(x + a - s) f(s, y(s)) ds, \tag{4.10}$$

where $z(x)$ is the solution of (4.2c) such that $z(a) = y(a) = y_0$ and Φ is the matrix solution of $\Phi' = A\Phi$ with $\Phi(a) = I$, the identity matrix. We know that any solution $z(x)$ of (4.2c) satisfying $z(a) = y_0$ can be expressed as $z(x) = \Phi(t)y_0$.

Now, since all the solutions of (4.2c) are bounded, let

$$c_1 = \max\{\sup_{t\geq a} \|z(t)\|, \ \sup_{t\geq a} \|\Phi(t)\|\}.$$

Hence, from (4.10), we obtain

$$\begin{aligned} \|y(x)\| &\leq c_1 + c_1 \int_a^x \|f(s, y(s))\| ds \\ &\leq c_1 + c_1 \int_a^x (h(s)\|y(s)\| + \|f(s, 0)\|) ds \\ &\leq c_1 + c_1 \int_a^x h(s)\|y(s)\| + c_1 \int_a^\infty \|f(s, 0)\| ds \end{aligned}$$

Therefore

$$\|y(x)\| \leq c_1^* + c_1 \int_a^x h(s)\|y(s)\|ds \tag{4.11}$$

where $c_1^* = c_1 + c_1 \int_a^\infty \|f(s,0)\|ds$, which is finite according to our hypothesis. Now, by applying Grownwall inequality to equation (4.11) we get, for all $x \geq a$

$$\begin{aligned} \|y(x)\| &\leq c_1^* \exp[c_1 \int_a^x h(s)ds] \\ &\leq c_1^* \exp[c_1 \int_a^\infty h(s)ds] \end{aligned}$$

According to our hypothesis : $\int_a^\infty h(s)ds < \infty$, so it follows that all the solutions of (4.1c) are also bounded.

□

Lemma 4.3 *Let B be a constant matrix of the form*

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

where B_1, B_2 are square matrices such that the real parts of the eigenvalues of B_1 are negative while the real parts of those of B_2 are zero. Moreover, suppose all the eigenvalues of B_2 are simple. Let $f : [a,\infty) \times R^n \to R^n$ be continuous and such that

$$\int_a^\infty \|f(t,0)\|dt < \infty \quad \text{and} \quad \|f(x,y) - f(x,\bar{y})\| \leq h(x)\|y - \bar{y}\|$$

with $h(t)$ bounded on $[a,\infty)$ and $\int_a^\infty h(t)dt < \infty$. Then the systems $\phi' = B\phi$ and $\psi' = A\psi + f(x,\psi)$ are asymptotically equivalent by means of a homeomorphism between the sets of their respective solutions where the topology is that of uniform convergence on $[a,\infty)$.

Proof: We know that the fundamental matrix $\Phi(t)$ corresponding to the linear system $\phi' = B\phi$ is given by

$$\Phi(x) = e^{xB} = I + xB + \frac{x^2B^2}{2!} + \frac{x^3B^3}{3!} + \ldots.$$

Because of the special form of B, $\Phi(x)$ has the following form:

$$\Phi(x) = \begin{pmatrix} Z_1(x) & 0 \\ 0 & Z_2(x) \end{pmatrix},$$

where $Z_1(x)$ consists of decreasing exponential functions and $Z_2(x)$ contains constants and complex exponentials.

To this form of Φ, there correspond two supplementary projections P_1 and P_2 of R^n such that

$$\Phi(x)P_1 = \begin{pmatrix} Z_1(x) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(x)P_2 = \begin{pmatrix} 0 & 0 \\ 0 & Z_2(x) \end{pmatrix}.$$

Since

$$\|\Phi(x)P_1\Phi^{-1}(t)\| = \|Z_1(x)Z_1^{-1}(t)\| = \|Z_1(x-t)\|$$

and

$$\|\Phi(x)P_2\Phi^{-1}(t)\| = \|Z_2(x)Z_1^{-1}(t)\| = \|Z_2(x-t)\|$$

and Z_1, Z_2 consist of decreasing exponential functions, constants and complex exponentials respectively, it follows that there exists $K > 0$ such that

$$\|\Phi(x)P_1\Phi^{-1}(t)\| \leq K \quad \text{for} \quad 0 \leq t \leq x,$$

$$\|\Phi(x)P_2\Phi^{-1}(t)\| \leq K \quad \text{for} \quad 0 \leq x \leq t.$$

Moreover,

$$\lim_{x\to\infty} \Phi(x)P_1 = 0.$$

From lemma 4.2, we also know that all the solutions of the two systems (4.1c), (4.2c) are bounded. Hence all the hypotheses of the theorem are satisfied and it follows that the two given systems are asymptotically equivalent.

□

Corollary 4.1 *Let A be a constant $n \times n$ matrix whose eigenvalues have non-positive real part; suppose that the eigenvalues with real part zero are simple. Let $f : [a, \infty) \times R^n \to R^n$ be continuous and such that*

$$\int_a^\infty \|f(t,0)\|dt < \infty \quad \textit{and} \quad \|f(x,y) - f(x,\bar{y})\| \leq h(x)\|y - \bar{y}\|$$

with $h(t)$ bounded on $[a,\infty)$ and $\int_a^\infty h(t)dt < \infty$. Then the systems $z' = Az$ and $y' = Ay + f(x,y)$ are asymptotically equivalent by means of a homeomorphism between the sets of their respective solutions where the topology is that of uniform convergence on $[a,\infty)$.

Proof: By virtue of the theorem 4.2 on Jordan canonical form, there exists an invertible matrix T such that $T^{-1}AT$ is of the following form:

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are square matrices such that the real parts of the eigenvalues of A_1 are negative while the real parts of those of A_2 are zero. So, by making the transformations $z = T\phi$, and $y = T\psi$ our systems (4.2c) and (4.1c) becomes

$$\phi' = \tilde{A}\phi \tag{4.2c$'$}$$

and

$$\psi(t)' = \tilde{A}\psi + g(x,\psi) \tag{4.1c$'$}$$

respectively; where $g(x,\psi) = T^{-1}f((x,T\psi)$. Since

$$\begin{aligned} \int_a^\infty \|g(x,0)\|dt &= \int_a^\infty \|T^{-1}f(x,T0)\|dt \\ &\leq \int_a^\infty \|T^{-1}\|\|f(x,0)\|dt \\ &= \|T^{-1}\| \int_a^\infty \|f(x,0)\|dt < \infty \end{aligned}$$

and

$$\begin{aligned} \|g(x,\psi) - g(x,\overline{\psi})\| &= \|T^{-1}f(x,T\psi) - T^{-1}f(x,T\overline{\psi})\| \\ &\leq \|T^{-1}\|\|f(x,T\psi) - f(x,T\overline{\psi})\| \\ &\leq h(x)\|T^{-1}\|\|T\psi - T\overline{\psi}\| \\ &\leq h(x)\|T^{-1}\|\|T\|\|\psi - \overline{\psi}\| \\ &= \tilde{h}(x)\|\psi - \overline{\psi}\| \end{aligned}$$

by the above lemma, $(4.2c)'$ and $(4.1c)'$ are asymptotically equivalent by means of a homomorphism S' between the sets $X_1'([a,\infty))$ and $X_2'([a,\infty))$ of their respective solutions. We claim that the systems $(4.2c)$, and $(4.1c)$ are asymptotically equivalent by means of the homeomorphism $S := TS'T^{-1}$. Clearly S is a homeomorphism since T is a nonsingular operator on R^n. On the other hand, if $z \in X_1([a,\infty))$, then we have

$$\begin{aligned} \|z(x) - Sz(x)\| &= \|TT^{-1}z(x) - STT^{-1}z(x)\| \\ &= \|T(T^{-1}z(x) - T^{-1}STT^{-1}z(x))\| \\ &\leq \|T\|\|(T^{-1}z)(x) - S'(T^{-1}z)(x))\| \end{aligned}$$

Since $T^{-1}z \in X_1'([a,\infty))$,

$$\|(T^{-1}z)(x) - S'(T^{-1}z)(x))\| \to 0 \quad \text{as} \quad x \to \infty$$

Consequently, $\|z(x) - Sz(x)\| \to 0$ as $x \to \infty$, as desired.

□

Before proving the next corollary of our theorem, we need a result which we state and prove as a lemma below.

Lemma 4.4 *The zero solution $z \equiv 0$ of $z' = Az$, where A is a constant $n \times n$ matrix, is stable if all the eigenvalues of A have negative real part.*

Proof: We know that any solution $\psi(t)$ of $z' = Az$ is of the form $\psi(t) = e^{At}\psi(0)$. Let $\phi_{ij}(t)$ be the ij element of the matrix e^{At}, and let $\psi_1^0, ..., \psi_n^0$ be the components of $\psi(0)$. Then the $i-th$ component of $\psi(t)$ is

$$\psi_i(t) = \phi_{i1}(t)\psi_1^0 + ... + \phi_{in}(t)\psi_n^0 = \sum_{j=1}^{n} \phi_{ij}(t)\psi_j^0.$$

Let $-\alpha_1$ be the largest of the real parts of the eigenvalues of A. By using lemma 4.1, for every number $-\alpha$, with $-\alpha_1 < -\alpha < 0$, we can find a number K such that $|\phi_{ij}(t)| \leq Ke^{-\alpha t}, \quad t \geq 0$. Consequently,

$$|\psi_i(t)| \leq \sum_{j=1}^{n} Ke^{-\alpha t}|\psi_j^0| = Ke^{-\alpha t}\sum_{j=1}^{n} |\psi_j^0|$$

for some positive constants K and α. Now, $|\psi_j^0| \leq \|\psi(0)\|$, where $\|.\|$ stands for maximum norm. Hence,

$$\begin{aligned} \|\psi(t)\| &= \max\{|\psi_1(t)|, ..., |\psi_n(t)|\} \\ &\leq nKe^{-\alpha t}\|\psi(0)\|. \end{aligned}$$

Now, let $\epsilon > 0$ be given. Let's choose $\delta(\epsilon) = \epsilon/nK$. Then $\|\psi(t)\| < \epsilon$ if $\|\psi(0)\| < \delta(\epsilon)$ and $t \geq 0$, since

$$\|\psi(t)\| \leq nKe^{-\alpha t}\|\psi(0)\| < nK.\epsilon/nK = \epsilon.$$

Consequently, the zero solution $z(t) \equiv 0$ is stable.

□

Remark: Indeed the above lemma proves that every solution $\phi(t)$ of (4.2c) is stable. To see this, let $\psi(t)$ be any solution of (4.2c). We note that $z(t) := \phi(t) - \psi(t)$ is again a solution of (4.2c). Now, $z(t) \equiv 0$ is stable implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|\phi(0) - \psi(0)\| < \delta \quad \Rightarrow \quad \|\phi(t) - \psi(t)\| < \epsilon \text{ for all } t \geq 0.$$

Consequently, every solution $\phi(t)$ of (4.2c) is stable.

Now we are ready to prove the second corollary of our theorem.

Corollary 4.2 *Let A be a constant $n \times n$ matrix whose eigenvalues are all negative and let $f : [a, \infty) \times R^n \to R^n$ be a continuous function such that*

$$f(t, 0) \equiv 0 \quad \textit{and} \quad \|f(x, y) - f(x, \bar{y})\| \leq h(x)\|y - \bar{y}\|$$

with $h(t)$ bounded on $[a, \infty)$ and $\int_a^\infty h(t)dt < \infty$. Then $y \equiv 0$ is stable for $y' = Ay + f(x, y)$.

Proof: We note from the proof of lemma 4.3 that the conditions of the theorem 4.1 are satisfied with $P_2 = 0$. Let K be the constant determined in the proof of lemma 4.3, and let $c \geq a$ be such that

$$2K \int_c^\infty h(t)\, dt < 1.$$

Let $x_0 \geq c$. If we apply lemma 4.3 on the interval $[x_0, \infty)$, we obtain a homeomorphism S from the set of solutions of (4.2c) onto the set of solutions of (4.1c) with the topology of the uniform convergence on $[x_0, \infty)$. Since $P_2 = 0$, from (4.6) we have

$$z(x_0) = Sz(x_0) \tag{4.12}$$

Since S is continuous at $z(x) \equiv 0$, given $\epsilon_0 > 0$ there exists $\epsilon > 0$ such that

$$\|\tilde{z}\|_\infty \leq \epsilon_0 \quad \Rightarrow \quad \|S\tilde{z}\|_\infty < \epsilon \tag{4.13}$$

Now, by lemma 3.4, since the zero solution $z(x) \equiv 0$ of (4.2c) is stable, for any $\epsilon_0 > 0$, there exists $\delta > 0$ such that, for any solution $\tilde{z}(x)$ of (4.2c)

$$\|\tilde{z}(x_0)\| < \delta \quad \Rightarrow \quad \|\tilde{z}(x)\| < \epsilon_0 \text{ for all } x \geq x_0 \tag{4.14}$$

Now we'll easily see that $y(x) \equiv 0$, which is a solution of (4.1c) because of the hypothsis $f(t, 0) = 0$, is stable. Let $\tilde{y}(x)$ be any solution of (4.1c) satisfying $\|\tilde{y}(x_0)\| < \delta$. Set $\tilde{z} = S^{-1}\tilde{y}$. From (4.12) we have $\tilde{y}(x_0) = S\tilde{z}(x_0) = \tilde{z}(x_0)$. So, from (4.14), we obtain

$$\begin{aligned}
\|\tilde{y}(x_0)\| = \|\tilde{z}(x_0)\| < \delta \quad &\Rightarrow \quad \|\tilde{z}(x)\| < \epsilon_0 \text{ for all } x \geq x_0 \\
&\Rightarrow \quad \|S(\tilde{z})\|_\infty < \epsilon \quad \text{by (4.13)} \\
&\Rightarrow \quad \|\tilde{y}\|_\infty < \epsilon \\
&\Rightarrow \quad \|\tilde{y}(x)\| < \epsilon \text{ for all } x \geq x_0
\end{aligned}$$

as desired.

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